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Darboux Problem for the Third-Order Bianchi Equation

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Abstract—The existence and uniqueness of the solution of the Darboux problem are proved. The solution of the Darboux problem is constructed in terms of a function similar to the Riemann–Hadamard function.

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The Darboux problem for a second-order hyperbolic equation with two independent variables has been considered by many authors. For example, let us mention [1, pp. 228–233], [2]–[7].

In the present paper, we study the existence and uniqueness of the solution of the Darboux problem and construct the corresponding Riemann–Hadamard function for the third-order Bianchi equation studied, for example, in [8]–[13]. Darboux-type problems of the Bianchi equation were considered in some particular cases in [9], [11].

1. The equation

$$\begin{aligned} L(u) \equiv u_{xyz} &+ a_{110}(x,y,z)u_{xy} + a_{101}(x,y,z)u_{xz} + a_{011}(x,y,z)u_{yz} + a_{100}(x,y,z)u_x \\ &+ a_{010}(x,y,z)u_y + a_{001}(x,y,z)u_z + a_{000}(x,y,z)u = f(x,y,z) \end{aligned} \tag{1}$$

has been called the *third-order Bianchi equation* by several authors [12, p. 6], [13], [14].

We define the function class $C^{(k,l,m)}$ as follows: a function u belongs to the class $C^{(k_1,k_2,k_3)}$ if its continuous derivatives $\partial^{r_1+r_2+r_3}u/\partial x^{r_1}\partial y^{r_2}\partial z^{r_3}$, $r_i = 0,\dots,k_i$, exist. A solution of class $C^{(1,1,1)}$ is said to be regular. Let D be the domain bounded by the planes $x = 0$ (AA_1O_1O), $y = 0$ (AOB), $y = y_0 > 0$ $(A_1O_1B_1)$, $z = x$ (OBB_1O_1), $z = z_0 > 0$ (see the figure). We assume that the coefficients of Eq. (1) satisfy the smoothness conditions $a_{ijk} \in C^{(i,j,k)}(\overline{D})$. We let X, Y, and T denote the faces of D for $x = 0$, $y = 0$, and $z = x$, respectively.

Darboux Problem. *In the domain D, to determine the regular solution of Eq.* (1) *satisfying the boundary conditions*

$$\begin{gathered} u|_{\overline{X}} = \varphi_1(y,z), \qquad u|_{\overline{Y}} = \varphi_2(x,z), \qquad u|_{\overline{T}} = \psi(x,y), \\ \varphi_1(y,0) = \psi(0,y), \qquad \varphi_2(x,x) = \psi(x,0), \qquad \varphi_1(0,z) = \varphi_2(0,z), \\ \varphi_1 \in C^{(1,1)}(\overline{X}), \qquad \varphi_2 \in C^{(1,1)}(\overline{Y}), \qquad \psi \in C^{(1,1)}(\overline{T}). \end{gathered} \tag{2}$$

Let us prove the existence and uniqueness of the solution of the Darboux problem. For this, we use the well-known formula for the solution of the Goursat problem [8], [12, p. 28] as a representation of an arbitrary regular solution of Eq. (1). From this formula we derive the Volterra integral equation of the second kind which we use to pose the Goursat condition $u(x,y,z_0)$ on the plane $z = z_0$ and note that the existence and uniqueness of the solution of this problem imply the existence and uniqueness of the solution of the Darboux problem (1), (2).

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